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Some convergence theorems of implicit iterative process for nonexpansive mappings in Banach spaces

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Abstract. *The purpose of this paper is to study the convergence problem of implicit iteration process for a finite family of nonexpansive mappings in Banach spaces. The results presented in this paper not only generalize some recent results but also give an affirmative answer to the open question suggested by Xu and Ori [11].*

Key words: *implicit iteration process, finite family of nonexpansive mappings, common fixed point*

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1. Introduction and preliminaries

Throughout this paper, we assume that E is a real Banach space and $F(T)$ is the set of fixed points of mapping T.

In 2001, Xu and Ori [11] considered the following interesting problem:

Let H be a Hilbert space, C a nonexpansive closed convex subset of H and let $\{T_1, T_2, \cdots .T_N\}$ be N nonexpansive mappings from C into C such that

$$F := \cap_{i=1}^{N} F(T_i) \neq \emptyset.$$

Let $\{t_n\}$ be a real sequence in $(0,1)$, $x_0 \in C$ any given point and $\{x_n\}$ the sequence defined by

$$\begin{aligned}
x_1 &= t_1 x_0 + (1-t_1)T_1 x_1, \\
x_2 &= t_2 x_1 + (1-t_2)T_2 x_2, \\
&\vdots \\
x_N &= t_N x_{N-1} + (1-t_N)T_N x_N, \\
x_{N+1} &= t_{N+1} x_N + (1-t_{N+1})T_{N+1} x_{N+1}, \\
&\vdots
\end{aligned}$$

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which can be written in the following compact form:

$$x_n = t_n x_{n-1} + (1 - t_n) T_n x_n \ \ \forall \ \ n \geq 1. \tag{1}$$

where $T_k = T_{k(modN)}$. Under some additional condition, the authors proved that this sequence $\{x_n\}$ weakly converges to a common fixed point of $\{T_1, T_2, \cdots, T_N\}$ in H.

The purpose of this paper is to study the strong convergence problem of an implicit iterative sequence (1) converging to a common fixed point in Banach space. The results presented in this paper not only generalize and extend the corresponding results of Xu and Ori [11], Wittmann [9], Reich [6], and Shioji and Takahashi [5] but also give an affirmative answer to the open question suggested by Xu and Ori in [11].

For the sake of convenience, we first recall some definitions and notations.

Definition 1. *A Banach space E is said to be uniformly convex, if the modulus of convexity of E:*

$$\delta_E(\epsilon) = \inf\{1 - \frac{1}{2}||x + y|| : ||x|| = ||y|| = 1, ||x - y|| = \epsilon\} > 0, \ \ \forall 0 < \epsilon \leq 2.$$

Definition 2. *Let C be a closed subset of a Banach space E. A mapping $T : C \to C$ is said to be semi-compact, if for any sequence $\{x_n\}$ in C such that $||x_n - Tx_n|| \to 0 \ \ (n \to \infty)$, then there exists a subsequence $\{x_{n_i}\} \subset \{x_n\}$ such tht $x_{n_i} \to x^* \in C$.*

Lemma 1 [see [10]]. *Let $p > 1$ and $r > 0$ be two positive real numbers. Then a Banach space E is uniformly convex, if and only if there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$||\lambda x + (1 - \lambda)y||^p \leq \lambda||x||^p + (1 - \lambda)||y||^p - \omega_p(\lambda) g(||x - y||)$$

for all $x, y \in B(0, r)$ and $\lambda \in [0, 1]$,where $B(0, r)$ is the closed ball of E with center zero and radius r and

$$\omega_p(\lambda) = \lambda^p(1 - \lambda) + \lambda(1 - \lambda)^p.$$

2. Main results

Theorem 1. *Let C be a nonempty closed convex subset of a Banach space E and $\{T_1, T_2, \cdots, T_N\}$ be N nonexpansive mappings from C into C such that $F = \cap_{i=1}^N F(T_i) \neq \emptyset$. If the sequence $\{t_n\}$ appeared in (1) satisfies the following condition:*

(α) there exists a positive number $s \in (0, 1)$ such that $t_n \in (0, 1 - s)$

then the implicit iterative sequence $\{x_n\}$ defined by (1) strongly converges to a common fixed point $p \in F$, if and only if

$$\liminf_{n \to \infty} d(x_n, F) = 0, \tag{2}$$

where $d(x, F)$ denotes the distance of x to the set F.

Proof. The necessity of condition (2) is obvious. Next, we prove the sufficiency of *Theorem 1.* For any given $p \in F$ it follows from (1) that

$$\begin{aligned} ||x_n - p|| &= ||t_n x_{n-1} + (1 - t_n) T_n x_n - p|| \\ &\leq t_n ||x_{n-1} - p|| + (1 - t_n) ||T_n x_n - p|| \\ &\leq t_n ||x_{n-1} - p|| + (1 - t_n) ||x_n - p||. \end{aligned}$$

Since $t_n \in (0, 1)$, simplifying we have

$$||x_n - p|| \leq ||x_{n-1} - p||, \ \ \forall\, p \in F \ \ n \geq 1. \tag{3}$$

Therefore we have

$$d(x_n, F) \leq d(x_{n-1}, F), \ \ \forall n \geq 1.$$

By condition (2), we know that

$$\lim_{n \to \infty} d(x_n, F) = 0. \tag{4}$$

Next, we prove that $\{x_n\}$ is a Cauchy sequence in C. In fact, for any positive integers m, n, from (3) we have

$$||x_{n+m} - p|| \leq ||x_n - p||, \ \ \forall p \in F. \tag{5}$$

Since $\lim_{n \to \infty} d(x_n, F) = 0$, for any given ϵ, there exists a positive integer n_0 such that $d(x_n, F) < \frac{\epsilon}{3}$. Hence there exists a point $p_1 \in F$ such that $||x_{n_0} - p_1|| \leq \frac{\epsilon}{2}$. Therefore from (5) we have

$$\begin{aligned} ||x_{n+m} - x_n|| &\leq ||x_{n+m} - p_1|| + ||x_n - p_1|| \\ &\leq ||x_{n_0} - p_1|| + ||x_{n_0} - p_1|| < \epsilon, \ \ \forall \ n \geq n_0, \ \ m \geq 1. \end{aligned}$$

This implies that $\{x_n\}$ is a Cauchy sequence in C. By the closedness of C, we can assume that $x_n \to p \in C$.

Finally, we prove that p is a common fixed point of $\{T_1, T_2, \cdots, T_N\}$. In fact, for any $i \in \{1, 2, \cdots, N\}$ from (1) we have

$$x_{kN+i} = t_{kN+i} x_{kN+i-1} + (1 - t_{kN+i}) T_{kN+i} x_{kN+i}, \ \ \forall \ k \geq 0.$$

Simplifying we have

$$\begin{aligned} x_{kN+i} &= \frac{t_{kN+i}}{1 - t_{kN+i}} (x_{kN+i-1} - x_{kN+i}) + T_{kN+i} x_{kN+i} \\ &= \frac{t_{kN+i}}{1 - t_{kN+i}} (x_{kN+i-1} - x_{kN+i}) + T_i x_{kN+i} \ \ \forall \ k \geq 0. \end{aligned} \tag{6}$$

Letting $k \to \infty$ and taking the superior limit in (6) and noting the condition (α) we have

$$p = T_i p, \ \ \forall i = 1, 2, \cdots, N.$$

This completes the proof of *Theorem 1.* □

Theorem 2. *Let E be a real uniformly convex Banach space and C a bounded closed convex subset of E. Let $\{T_i, i = 1, 2, \cdots, N\}$ be N nonexpansive mappings from C into C such that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. If the following conditions are satisfied:*

(i) *there exists some* $T \in \{T_i, i = 1, 2, \cdots, N\}$ *(without loss of generality we can assume that it is* T_1*) which is semi-compact;*

(ii) *The condition* (α) *in* Theorem 1 *is satisfied.*

then the implicit iterative sequence $\{x_n\}$ *defined by (1) strongly converges to a common fixed point* $p \in F$.

Proof. Since C is bounded, take $r > 0$ such that $C \subset B(0, r)$, where $B(0, r)$ is the closed ball of E with center zero and radius r. By *Lemma 1* with $p = 2$, for any $p \in F$ we have

$$\begin{aligned}||x_n - p||^2 &\leq t_n||x_{n-1} - p||^2 + (1 - t_n)||T_n x_n - p||^2 - t_n(1 - t_n)g(||x_{n-1} - T_n x_n||)\\ &\leq t_n||x_{n-1} - p||^2 + (1 - t_n)||x_n - p||^2 - t_n(1 - t_n)g(||x_{n-1} - T_n x_n||).\end{aligned}$$

Simplifying and noting $t_n \in (0, 1)$, we have

$$||x_n - p||^2 \leq ||x_{n-1} - p||^2 - (1 - t_n)g(||x_{n-1} - T_n x_n||).$$

By condition (ii), we have

$$s \cdot g(||x_{n-1} - T_n x_n||) \leq (1 - t_n)g(||x_{n-1} - T_n x_n||) \leq ||x_{n-1} - p||^2 - ||x_n - p||^2.$$

Therefore, for any positive integer m, we have

$$s \sum_{n=1}^{m} g(||x_{n-1} - T_n x_n||) \leq ||x_0 - p|| - ||x_m - p|| \leq ||x_0 - p||.$$

Letting $m \to \infty$, we have

$$s \sum_{n=1}^{\infty} g(||x_{n-1} - T_n x_n||) \leq ||x_0 - p|| < \infty.$$

This implies that

$$g(||x_{n-1} - T_n x_n||) \to 0 \ \ (n \to \infty). \tag{7}$$

Since g is a continuous and strictly increasing function with $g(0) = 0$, it follows from (7) that

$$||x_{n-1} - T_n x_n|| \to 0 \ \ (n \to \infty). \tag{8}$$

From (1) and (8), we have

$$||x_n - x_{n-1}|| = (1 - t_n)||x_{n-1} - T_n x_n|| \to 0.$$

Therefore we have

$$||x_n - T_n x_n|| \leq ||x_n - x_{n-1}|| + ||x_{n-1} - T_n x_n|| \to 0 \ \ (n \to \infty),$$

and so

$$||x_n - x_{n+i}|| \to 0 \ \ (n \to \infty) \ \ \forall i = 1, 2, \cdots N.$$

For any $i = 1, 2, \cdots, N$ we have

$$\begin{aligned} ||x_n - T_{n+i}x_n|| &\leq ||x_n - x_{n+i}|| + ||x_{n+i} - T_{n+i}x_{n+i}|| \\ &\quad + ||T_{n+i}x_{n+i} - T_{n+i}x_n|| \\ &\leq 2||x_n - x_{n+i}|| + ||x_{n+i} - T_{n+i}x_{n+i}||. \end{aligned}$$

Hence we have

$$\lim_{n\to\infty} ||x_n - T_{n+i}x_n|| = 0 \ \ \forall i = 1, 2, \cdots, N. \tag{9}$$

Taking $i = 1$, from (9) we have

$$\lim_{k\to\infty} ||x_{kN} - T_{kN+1}x_{kN}|| = \lim_{k\to\infty} ||x_{kN} - T_1 x_{kN}|| = 0. \tag{10}$$

By the assumption, T_1 is semi-compact, there exists a subsequence $\{x_{k_iN}\} \subset \{x_{kN}\}$ such that $x_{k_iN} \to q \in C$ as $k_i \to \infty$. Hence from (9) for any $j = 1, 2, \cdots, N$ we have

$$||q - T_j q|| = \lim_{k_i\to\infty} ||x_{k_iN} - T_{k_iN+j}x_{k_iN}|| = 0.$$

This implies that q is a common fixed point in F. Therefore we have $\liminf d(x_n, F) = 0$. By *Theorem 1* we know that x_n converges strongly to a common fixed point in F. This completes the proof of *Theorem 2*. □

Remark 1. Theorem 1 *and* Theorem 2 *not only generalize and improve the corresponding in [5, 6, 9, 11], but also give an affirmative answer to the open question suggested by Xu and Ori [11]: " It is yet unclear what assumptions on the mappings* $\{T_1, T_2, \cdots, T_N\}$ *and/or the parameters* $\{t_n\}$ *are sufficient to guarantee the strong convergence of the sequence* $\{x_n\}$ *defined by (1)."*

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